Relative Weak Injectivity for Operator Systems

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Outline

- Operator systems and tensor products
- Relative Weak Injectivity For C*-algebras
- Relative Weak Injectivity For Operator Systems
  - WEP and WRI for operator systems
  - C*-systems and (c,max)-nuclearity
  - Relative double commutant injectivity
  - Quasi-nuclearity

Concrete Operator Systems

A **(concrete) operator system** $\mathcal{S}$ is a unital, $*$-closed subspace of $B(\mathcal{H})$ together with the induced matricial order structure.

Looking at this way:

(1) $\mathcal{S}$ has an involution $*$ (i.e. self-adjoint idempotent), therefore $\mathcal{S}$ has self-adjoint elements ($s = s^*$), denoted by $\mathcal{S}_{sa}$

(2) $\mathcal{S}$ has positive elements: $\mathcal{S}^+ = \mathcal{S} \cap B(\mathcal{H})^+$, which forms a cone,

(3) $M_n(\mathcal{S})$, $n \times n$ matrices with entries belongs to $\mathcal{S}$, has also positive elements with the identification

$$M_n(B(\mathcal{H})) \cong B(\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}),$$

(4) the cones $\{M_n(\mathcal{S})^+\}_{n=1}^{\infty}$ are *compatible*: for any $n \times m$ matrix A we have

$$A^* M_n(\mathcal{S})^+ A \subseteq M_m(\mathcal{S})^+$$

(5) $\mathcal{S}$ has an matricial Archimedean order unit I:

order unit: $\forall x \in \mathcal{S}_{sa}, \exists \alpha \in \mathbb{R}^+$ with $\alpha I + x \in \mathcal{S}^+$.

Archimedean: $\forall \varepsilon > 0 \;\; x + \varepsilon I \geq 0 \Rightarrow x \geq 0$.

matricial: so $\begin{bmatrix} I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I \end{bmatrix}$ is Archimedean order unit for $M_n(\mathcal{S})$.

Abstract Operator Systems, Morphisms

An (abstract) **operator system** $\mathcal{S} = (\mathcal{S}, \{M_n(\mathcal{S})^+\}_{n=1}^{\infty}, 1)$ where

(1) $\mathcal{S}$ is a $*$-vector space,

(2) $\{M_n(\mathcal{S})^+\}_{n=1}^{\infty}$ is a cone of strict, compatible collection of positive elements with $M_n(\mathcal{S})^+ \subseteq M_n(\mathcal{S})_{sa}$,

(3) 1 is a matricial Archimedean order unit.

A linear map $\varphi : \mathcal{S} \to \mathcal{T}$ is called **positive** if $\varphi(\mathcal{S}^+) \subseteq \mathcal{T}^+$, **completely positive** if $\varphi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ is positive for all n, **unital** if $\varphi(1) = 1$.

φ is said to be a **ucp** map if it's unital and completely positive.

φ is said to be a **unital complete order isomorphism** if φ and φ^{-1} are both ucp maps.

Theorem. (Choi, Effros, '77) Every (abstract) operator system is unitally completely order isomorphic to a concrete operator system.

Tensor Products

We endow the algebraic tensor product $\mathcal{S} \otimes \mathcal{T}$ with a matricial order structure $\{C_n\}_{n=1}^{\infty}$ such a way that it is an operator system with unit $1 \otimes 1$. Moreover, the matricial order structure has certain compatibility properties with the underlying structures of $\mathcal{S}$ and $\mathcal{T}$. For example we require

$$M_n(\mathcal{S})^+ \otimes M_m(\mathcal{T})^+ \subseteq C_{nm}$$

Also, if $\varphi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are ucp maps then

$$\varphi \otimes \psi : \mathcal{S} \otimes \mathcal{T} \to M_n \otimes M_m \cong M_{nm}$$

must also be a ucp map.

Operator systems possess many different tensor products and the set of all tensor products form a partially ordered set with a minimal and maximal one:

$$\otimes_{\rm min} \;\leq \otimes_{\rm e} \;\leq\; \otimes_{\rm el} \;,\; \otimes_{\rm er} \;,\; \otimes_{\rm ess} \;\leq\; \otimes_{\rm c} \;\leq \otimes_{\rm max}$$

Today we will need: $\otimes_{\rm min} \;\leq\; \otimes_{\rm er} \;\leq\; \otimes_{\rm c} \;\leq \otimes_{\rm max}.$

Tensor Products 2

The minimal tensor product is the spatial one: for $\mathcal{S} \subseteq B(\mathcal{H})$ and $\mathcal{T} \subseteq B(\mathcal{K})$ the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ arising from the inclusion $B(\mathcal{H} \otimes_2 \mathcal{K})$ can be declared as the minimal tensor product.

To define maximal tensor products we first introduce

$$D_n^{\max} = \{A^*(X \otimes Y)A : \ X \in M_k(\mathcal{S})^+, \ Y \in M_m(\mathcal{T})^+,$$

$$A \text{ is } km \times n \text{ matrix}, k, m \in \mathbb{N}\}.$$

Setting $C_n^{\max} = \overline{D_n^{\max}}$, the closure of $D_n^{\max}$ relative to order norm, we obtain a matricial order structure $\{C_n^{\max}\}_{n=1}^{\infty}$ on $\mathcal{S} \otimes \mathcal{T}$. The resulting operator system is denoted by $\mathcal{S} \otimes_{\max} \mathcal{T}$.

The right injective tensor products: for $\mathcal{T} \subseteq B(\mathcal{H})$ and $\mathcal{S}$ we decleare

$$\mathcal{S} \otimes_{\text{er}} \mathcal{T} :\subseteq \mathcal{S} \otimes_{\max} B(\mathcal{H}).$$

Commuting tensor product of $\mathcal{S}$ and $\mathcal{T}$ can be defined as follows

$$\mathcal{S} \otimes_{\text{c}} \mathcal{T} :\subseteq C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}).$$

Nuclearity

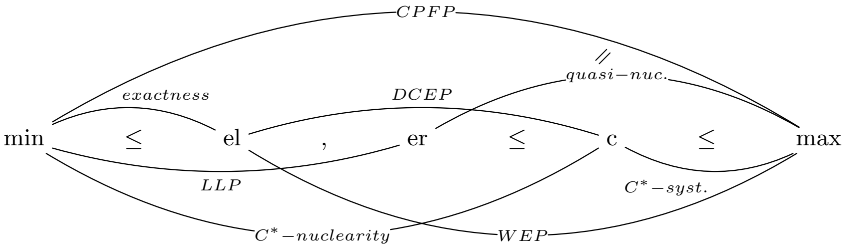
For operator system tensor products α and β we write $\alpha \leq \beta$ if for every operator systems $\mathcal{S}$ and $\mathcal{T}$ the canonical map

$$\mathcal{S} \otimes_{\beta} \mathcal{T} \to \mathcal{S} \otimes_{\alpha} \mathcal{T} \text{ is completely positive.}$$

$\mathcal{S}$ is said to be (α, β)-nuclear if for every operator system $\mathcal{T}$ we have

$$\mathcal{S} \otimes_{\alpha} \mathcal{T} = \mathcal{S} \otimes_{\beta} \mathcal{T} \text{ canonically.}$$

$\mathcal{S}$ is called nuclear if it is (min,max)-nuclear.



Weak expectation property (WEP)

C. Lance ('72) introduces following fundamental nuclearity property:

Definition. A unital C*-algebra $\mathcal{A}$ has **WEP** if for every faithful embedding $\mathcal{A} \subset B(\mathcal{H})$ there is a ucp map $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ with $\varphi(a) = a$ for every $a \in \mathcal{A}$ and image$(\varphi) \subset A''$.

WEP has several different formulations and for von Neumann algebras it is equivalent to injectivity. As pointed out by Effros and Haagerup, for general C*-algebras, WEP is equivalent to approximate injectivity in matrix systems.

WEP is a non-commutative property and it coincides with categorical nuclearity in Banach spaces or Kadison spaces.

WEP is a nuclearity related property, in fact $\mathcal{A}$ has WEP if and only if

$$\mathcal{A} \otimes_{\max} \mathcal{C} \subset \mathcal{B} \otimes_{\max} \mathcal{C} \text{ for all } \mathcal{C} \text{ and } \mathcal{B} \supset \mathcal{A}.$$

Kirchberg's QWEP conjecture states that every unital C*-algebra is a quotient of a C*-algebra with WEP.

Relative weak injectivity

E. Kirchberg ('93) introduces the notion of w.r.i.

Definition. A unital C*-subalgebra $\mathcal{A}$ of $\mathcal{B}$ is said to be **w.r.i.** in $\mathcal{B}$ if the canonical inclusion of $\mathcal{A}$ into bidual von Neumann algebra $\mathcal{A}^{**}$ extends to a completely positive map on $\mathcal{B}$. Equivalently, for every Hilbert space $\mathcal{H}$, for every ucp map $\varphi : \mathcal{A} \to B(\mathcal{H})$, φ extends to a ucp map $\tilde{\varphi} : \mathcal{B} \to B(\mathcal{H})$ such that $image(\tilde{\varphi}) \subseteq \varphi(\mathcal{A})''$.

In this case $\mathcal{A}$ inherits all the nuclearity-related properties of $\mathcal{B}$: CPFP, LLP, WEP etc.

WRI is a nuclearity related property:

Theorem. (Kirchberg) The following are equivalent for $\mathcal{A} \subseteq \mathcal{B}$:

(1) $\mathcal{A}$ is w.r.i. in $\mathcal{B}$;

(2) $\mathcal{A} \otimes_{\max} \mathcal{C} \subseteq \mathcal{B} \otimes_{\max} \mathcal{C}$ for every C*-algebra $\mathcal{C}$;

(3) $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) \subseteq \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_\infty)$.

Tight Riesz Interpolation / Riesz Arveson Decomposition

Definition: Consider $\mathcal{A} \subset \mathcal{B}$. We say that $\mathcal{A}$ has tight (n,k)-Riesz interpolation property in $\mathcal{B}$, TR(n,k)-property in short, if self-adjoint elements $x_1, x_2, .., x_n$ and $y_1, y_2, ..., y_k$ of $\mathcal{A}$ interpolate in $\mathcal{B}$, meaning that there is a self-adjoint element b in $\mathcal{B}$ such that

$$x_1, x_2, .., x_n < b < y_1, y_2, ..., y_k,$$

then $x_1, x_2, .., x_n$ and $y_1, y_2, ..., y_k$ interpolate in $\mathcal{A}$.
Likewise, we say that $\mathcal{A}$ has the complete TR(n,k)-property in $\mathcal{B}$, if $M_p(\mathcal{A})$ has the TR(n,k)-property in $M_p(\mathcal{B})$ for every p.

Definition: Consider $\mathcal{A} \subset \mathcal{B}$. We say that $\mathcal{A}$ has (n,k)-Riesz-Arveson decomposition property in $\mathcal{B}$, if for $\mathcal{H} = \ell^2(\mathbb{N})$, every cp maps $\phi_i, \psi_j : \mathcal{A} \to B(\mathcal{H})$, $i = 1, ..., n$, $j = 1, ..., m$ with $\sum \phi_i = \sum \psi_j$ extends to cp maps $\tilde{\phi}_i, \tilde{\psi}_j : \mathcal{B} \to B(\mathcal{H})$ such that

$$\sum \tilde{\phi}_i = \sum \tilde{\psi}_j.$$

More on w.r.i. for C*-algebras

Theorem. (K. '16) The following are equivalent for C*-algebras $\mathcal{A} \subseteq \mathcal{B}$:

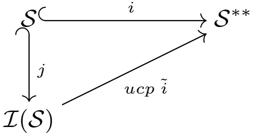
(1) $\mathcal{A}$ is w.r.i. in $\mathcal{B}$;

(2) $\mathcal{A}$ has the complete tight (n, k)-Riesz interpolation in $\mathcal{B}$ for every n, k;

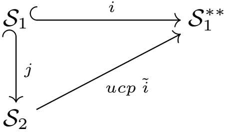
(3) $\mathcal{A}$ has the (n, k)-Riesz-Arveson decomposition property in $\mathcal{B}$ for all n, k.

Relative weak injectivity for operator systems

WEP: An operator system $\mathcal{S}$ is said to have **weak expectation property** if the canonical inclusion $\mathcal{S} \hookrightarrow \mathcal{S}^{**}$ decomposes through an injective object via ucp maps.



WRI: Consider $\mathcal{S}_1 \subseteq \mathcal{S}_2$, where $\mathcal{S}_1$ is an operator subsystem of $\mathcal{S}_2$. $\mathcal{S}_1$ is said be **relatively weakly injective** in $\mathcal{S}_2$ if the canonical inclusion $\mathcal{S}_1 \hookrightarrow \mathcal{S}_1^{**}$ extends to ucp map on $\mathcal{S}_2$:



WRI and nuclearity

Theorem. (K.) The following are equivalent for $\mathcal{S}_1 \subseteq \mathcal{S}_2$:

(1) $\mathcal{S}_1$ is w.r.i. in $\mathcal{S}_2$;

(2) $\mathcal{S}_1 \otimes_{\max} \mathcal{T} \subset \mathcal{S}_2 \otimes_{\max} \mathcal{T}$, for every operator system $\mathcal{T}$;

(3) for every n and null-subspace $J \subset M_n$ we have unital order embedding

$$\mathcal{S}_1 \otimes_{\max} (M_n/J) \subset \mathcal{S}_2 \otimes_{\max} (M_n/J);$$

(4) for all matrix systems $\mathcal{R}$, every ucp map $\varphi : \mathcal{S}_1 \to \mathcal{R}$ has a ucp extension $\tilde{\varphi} : \mathcal{S}_2 \to \mathcal{R}$;

(5) for every operator system $\mathcal{T}$, every ucp map $\varphi : \mathcal{S}_1 \to \mathcal{T}^{**}$ extends to a ucp map $\tilde{\varphi} : \mathcal{S}_2 \to \mathcal{T}^{**}$;

(6) every state φ on $\mathcal{S}_1$ has a state extension $\tilde{\varphi}$ on $\mathcal{S}_2$ such that if ψ is positive linear functional on $\mathcal{S}_1$ with $\psi \leq \varphi$, then ψ has positive extension $\tilde{\psi}$ on $\mathcal{S}_2$ with $\tilde{\psi} \leq \tilde{\varphi}$. Moreover, this can be achieved such a way that $\psi \mapsto \tilde{\psi}$ is a cp map from $[\varphi]$ to $[\tilde{\varphi}]$ for which the restriction is the ucp inverse;

(7) $\mathcal{S}_1^{**}$ is w.r.i. in $\mathcal{S}_2^{**}$, moreover the inclusion of $\mathcal{S}_1^{**}$ into $\mathcal{S}_2^{**}$ has a ucp inverse.

C*-systems / WEP

Definition. An operator system $\mathcal{S}$ for which the bidual operator system $\mathcal{S}^{**}$ has a structure of a C*-algebra is called a **C*-system**.

Theorem. (K.) The following are equivalent for $\mathcal{S}$:

(1) $\mathcal{S}$ is a C*-system;

(2) $\mathcal{S}$ is (c,max)-nuclear;

(3) $\mathcal{S}$ is w.r.i. in $C_u^*(\mathcal{S})$.

Theorem. The following are equivalent for $\mathcal{S}$:

(1) $\mathcal{S}$ has WEP;

(2) $\mathcal{S}$ is (el,max)-nuclear;

(3) for every n and null-subspace $J \subset M_n$ we have an order osmorphism

$$\mathcal{S} \otimes_{\min} (M_n/J) = \mathcal{S} \otimes_{\max} (M_n/J);$$

(4) For every n and matrix system $\mathcal{R} \subset M_n$, for every cp map $\varphi : \mathcal{R} \to \mathcal{S}$ and $\epsilon > 0$, there exists a cp map $\tilde{\varphi} : M_n \to \mathcal{S}$ such that $\|\tilde{\varphi}|_{\mathcal{R}} - \varphi\|_{cb} \leq \epsilon$.

Relative Double Commutant Injectivity

Consider $\mathcal{S}_1 \subset \mathcal{S}_2$. $\mathcal{S}_1$ is said to have r.d.c.i. in $\mathcal{S}_2$ if for every embedding $i : \mathcal{S} \hookrightarrow B(\mathcal{H})$, i extends to a ucp map $\tilde{i} : \mathcal{S}_2 \to B(\mathcal{H})$ such that $\mathrm{Im}(\tilde{i}) \subseteq \mathrm{Im}(i)''$.

Theorem. (A. Bhattacharya) The following are equivalent for $\mathcal{S}_1 \subseteq \mathcal{S}_2$:

(1) $\mathcal{S}_1$ has r.d.c.i. in $\mathcal{S}_2$;

(2) $\mathcal{S}_1 \otimes_c \mathcal{T} \subset \mathcal{S}_2 \otimes_c \mathcal{T}$, for every operator system $\mathcal{T}$;

(3) $\mathcal{S}_1 \otimes_{\max} \mathcal{A} \subset \mathcal{S}_2 \otimes_{\max} \mathcal{A}$, for every C*-algebra $\mathcal{A}$;

(4) We have a complete order embedding

$$\mathcal{S}_1 \otimes_{\max} C^*(\mathbb{F}_\infty) \subset \mathcal{S}_2 \otimes_{\max} C^*(\mathbb{F}_\infty).$$

(5) $C_u^*(\mathcal{S}_1)$ is w.r.i. in $C_u^*(\mathcal{S}_2)$.

(6) The canonical inclusion $i : \mathcal{S}_1 \hookrightarrow C_u^*(\mathcal{S}_1)^{**}$ extends to a ucp map on $\mathcal{S}_2$.

Setting $\mathcal{W}_6 = \{a_1, ..., a_6 : a_1 + a_2 + a_3 = a_4 + a_5 + a_6\} \subseteq \ell_6^\infty$

(7) $\mathcal{S}_1 \otimes_c \mathcal{W}_6^* \subseteq \mathcal{S}_2 \otimes_c \mathcal{W}_6^*$ completely order isomorphically.

Namioka and Phelp's test systems

We set $\mathcal{W}_{2n} = \{(a_i)_{i=1}^{2n} : \sum_{i=1}^{n} a_i = \sum_{i=n+1}^{2n} a_i\} \subseteq \ell_{2n}^{\infty}$.

Theorem. (Namioka, Phelp) TFAE for a Kadison function system $\mathcal{V}$:

(1) $\mathcal{V}$ is nuclear, that is, for every Kadison function system $\mathcal{W}$ we have

$$\mathcal{V} \otimes_{\varepsilon} \mathcal{W} = \mathcal{V} \otimes_{\pi} \mathcal{W};$$

(2) we have a canonical order isomorphism $\mathcal{V} \otimes_{\varepsilon} \mathcal{W}_4 = \mathcal{V} \otimes_{\pi} \mathcal{W}_4$.

Theorem. (K. '15) TFAE for a unital C*-algebra $\mathcal{A}$:

(1) $\mathcal{A}$ is nuclear, i.e., for every C*-algebra $\mathcal{B}$ we have $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ (equivalently for every operator system $\mathcal{S}$ we have $\mathcal{A} \otimes_{\min} \mathcal{S} = \mathcal{A} \otimes_{\max} \mathcal{S}$);

(2) $\mathcal{A} \otimes_{\min} \mathcal{W}_6 = \mathcal{A} \otimes_{\max} \mathcal{W}_6$ completely order isomorphically.

Theorem. (K. '18) A C*-system $\mathcal{S}$ is nuclear if and only if $\mathcal{S} \otimes_{\min} \mathcal{W}_6 = \mathcal{S} \otimes_{\max} \mathcal{W}_6$.

Quasi-nuclearity

Definition. An operator system $\mathcal{S}$ is called **quasi-nuclear** if for all $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we have $\mathcal{S} \otimes_{\max} \mathcal{T}_1 \subseteq \mathcal{S} \otimes_{\max} \mathcal{T}_2$.

Theorem. (K. '18) The following are equivalent for an operator system $\mathcal{S}$:

(1) $\mathcal{S}$ is nuclear;

(2) $\mathcal{S}$ is quasi-nuclear;

(3) $\mathcal{S}$ is (er,max)-nuclear;

(4) for every matrix system $\mathcal{R}$ we have $\mathcal{S} \otimes_{\min} \mathcal{R} = \mathcal{S} \otimes_{\max} \mathcal{R}$.

Question: if $\mathcal{S} \otimes_{\min} \mathcal{W}_6 = \mathcal{S} \otimes_{\max} \mathcal{W}_6$. can we conclude that $\mathcal{S}$ is nuclear?

More Questions

Setting $J = span\{(1,1,1,-1,-1,-1)\} \subset \ell_6^\infty$ we have a complete order isomorphism

$$\ell_6^\infty / J = \mathcal{W}_6^*.$$

For C*-algebras $\mathcal{A} \subseteq \mathcal{B}$ we have

$$\mathcal{A} \otimes_{\max} (\ell_6^\infty / J) \subseteq \mathcal{B} \otimes_{\max} (\ell_6^\infty / J)$$

implies that $\mathcal{A}$ is w.r.i. in $\mathcal{B}$, that is

$$\mathcal{A} \otimes_{\max} \mathcal{T} \subseteq \mathcal{B} \otimes_{\max} \mathcal{T}$$

for all operator system $\mathcal{T}$. Likewise, for operator systems $\mathcal{S}_1 \subseteq \mathcal{S}_2$ a canonical inclusion

$$\mathcal{S}_1 \otimes_{\mathrm{c}} (\ell_6^\infty / J) \subseteq \mathcal{S}_2 \otimes_{\mathrm{c}} (\ell_6^\infty / J)$$

implies that $\mathcal{S}_1$ has r.d.c.i. in $\mathcal{S}_2$.

Question: Let $\mathcal{S}_1 \subseteq \mathcal{S}_2$ be given. If we have a canonical inclusion

$$\mathcal{S}_1 \otimes_{\max} (\ell_6^\infty / J) \subseteq \mathcal{S}_2 \otimes_{\max} (\ell_6^\infty / J)$$

can we conclude that $\mathcal{S}_1$ has w.r.i. in $\mathcal{S}_2$?

THANKS!